Boundary behavior of conformal maps: first results

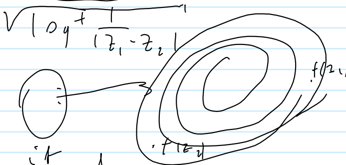
Friday, August 16, 2019 3:28 PM

Pre-cursor of the Extremal Length method was Length-Area Method, based on the following trivial estimate:

$\lambda(\Gamma) \geq \frac{(\inf_{\gamma\in\Gamma} \ell(\gamma\cap E))^2}{\text{Area}(E)}$ (take $\rho \equiv \chi_E$). So, if f is conformal, $\inf_{\gamma\in\Gamma} \ell(f(\gamma)\cap E) \leq (\lambda(\Gamma)\,\text{Area}(E))^{1/2}$.

If you know $\lambda(\Gamma)$, can estimate $\ell(f(\gamma)\cap E)$.

Thm (Wolf lemma). Let $f:\mathbb{D}\to\Omega$ - conformal, $\text{Area}(\Omega)<\infty$. Let $z_1, z_2 \in \mathbb{D}$. Then $\exists\, \beta$ - crosscut in Ω, separating z_1 and z_2 from 0, with $\ell(f(\beta)) \leq \frac{C(\text{Area}(\Omega))^{1/2}}{\sqrt{\log^+\frac{1}{|z_1-z_2|}}}$. $C = C_{abs}$.

Remarks. 1)  - $|z_1|$ can be very far from $f(z_2)$.

2) Not true if $A(\Omega)=\infty$.

3) $\frac{1}{2}$ is optimal, i.e. not true for $\frac{C}{\left(\log\frac{1}{|z_1-z_2|}\right)^{\theta}}$.

Pf. Can only consider $|z_1-z_2|<\frac{1}{10}$. Also, if $|z_1|<\frac{1}{2}$ or $|z_2|<\frac{1}{2}$, then by distortion thm, $|f(z_1)-f(z_2)| \leq C|f'(0)|\,|z_1-z_2| \leq 4C(\text{Area}(\Omega))^{1/2}|z_1-z_2|$ (by Köbe $\frac{1}{4}$, $f(\mathbb{D}) \supset \left(\frac{1}{4}|f'(0)|\right)\mathbb{D}+f(0)$).

Then  $\forall\, \Gamma$ - curves in $\{z\in\mathbb{D} : |z-z_1|<\frac{1}{2}\} =: A$ separating z_1 and z_2 from 0.

$\lambda(\Gamma) \leq \frac{2\pi}{\log\frac{1}{2|z_1-z_2|}}$ - extension rule.

So $\exists\, \beta\in f(\Gamma)$: $\ell(\beta) \leq \left(\frac{2\pi}{\log\frac{1}{2|z_1-z_2|}}\,\text{Area}\, f(A)\right)^{\frac{1}{2}} \leq \frac{(4\pi\,\text{Area}(\Omega))^{\frac{1}{2}}}{\sqrt{\log\frac{1}{|z_1-z_2|}}}$ - length-area.

Remark (Important). Can choose γ among circular arcs centered at z_1.

Thm (Lavrentiev). Let $I\subset\partial\mathbb{D}$ - arc, z_I - center of an arc orthogonal to I, β - crosscut in $\mathbb{D}$ with ends in ends of I. Then $\text{diam}\, f(\beta) \geq C_{abs}|I|\,|f'(z_I)|$.

Cor. 1 $\text{diam}\, f^*(I) \geq C|I|\,|f'(z_I)|$.

Cor. 2. $\text{diam}\, f(\beta) \geq C\,|I|^2|f'(0)|$

Pf (of Cor. 2). By distortion thm, for $f\in(S)$, $|f'(z_I)| \geq \frac{1-|z_I|}{(1+|z|)^3} \geq \frac{|I|}{8}$.

Now normalize.

Pf (of Thm). Normalize: $f\in(S)$ everything scales. Also, compose with Möbius to get $I=\mathbb{T}_+ = \{z\in\mathbb{T}, \text{Im}\, z>0\}$.

Need: $\text{diam}\, f(\beta) \geq C_{abs}>0$.

Let $\Delta := \frac{1}{2}\mathbb{D} = \{|z|<\frac{1}{2}\}$.

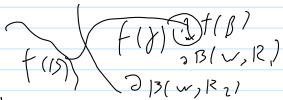
Two cases:

Case 1: $\beta\cap\Delta\neq\emptyset \Rightarrow \exists\, z\in\beta\cap\Delta \Rightarrow \text{diam}\, f(\beta) \geq \text{dist}(f(z), \partial\Omega) \geq \frac{1}{2}\frac{1}{4}|f'(z)| \geq \frac{1}{8}\frac{1-|z|}{(1+|z|)^3} \overset{|z|=\frac{1}{2}}{\geq} \frac{1}{128}$

Case 2: $\beta\cap\Delta=\emptyset$. $\forall\, \Gamma$ - curves in $\mathbb{D}\setminus\Delta$ from Δ to $\mathbb{T}_+$. $\forall\gamma\in\Gamma$, $\gamma\cap\beta\neq\emptyset$. $\lambda(\Gamma)$ is a finite constant (in fact, by extension, $\lambda(\Gamma)\leq\lambda(\Gamma^*)$, where Γ^* joins, so $\lambda(\Gamma)\leq\lambda(\Gamma^*) = \frac{\log 2}{\pi}$ - by symmetry).

Let $\text{diam}\, f(\beta) << 1$. Put $f(\beta)$ inside a disk $B(w,R)$, $R := \text{diam}\, f(\beta)$

Let $R_2 := \operatorname{dist}(f(B), f(S)) - \operatorname{diam} f(B) \geq \operatorname{dist}(\partial \mathbb{D}, f(B)) - 2\operatorname{diam} f(B) \overset{\text{Koebe}}{\geq}$
$\frac{1}{128} - \operatorname{diam} f(B) = \frac{1}{128} - R_1$, so $\forall \gamma \in f(\Gamma)$ intersects both $B(w, R_1)$ and $B(w, R_2)$:



so $\lambda(f(\Gamma)) \geq \frac{\log \frac{R_2}{R_1}}{2\pi} \geq$
$\frac{\log\left(\frac{1}{128 R_1} - 1\right)}{2\pi} \gg \lambda(\Gamma)$, provided R_1 is small enough - Contradiction! ∎

<u>Refinement</u> (w/o proof). $\exists C_{abs} > 0 : \forall f \in (S)$:
$z_1, z_2 \in \mathbb{D}$, γ - crosscut or closed curve separating $f(z_1), f(z_2)$ from 0, then
$\operatorname{diam}(\sigma) \geq C_{abs} |z_1 - z_2|^2$.

<u>Def</u>. (Mazurkiewicz metric) Fix $w_0 \in \Omega$
$\forall$ $\rho(w_1, w_2) := \inf \operatorname{diam} \sigma$, σ separates w_1, w_2 from w_0. Extend to Ω by continuity.
on $\Omega \setminus \{w_0\}$

<u>Thm</u>. $C|z_1 - z_2|^2 \leq \rho(f(z_1), f(z_2)) \leq \frac{C}{\sqrt{\log \frac{1}{|z_1 - z_2|}}}$ Can extend up to boundary!

Prime ends.